# Unitary and Asymptotic Behavior of Amplitudes in Non-Anticommutative Quantum Field Theory

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**Abstract**

The unitarity condition for scattering amplitudes in a non-anticommutative quantum field theory is investigated. The Cutkosky rules are shown to hold for Feynman diagrams in Euclidean space and unitarity of amplitudes can be satisfied. An analytic continuation of the diagrams to physical Minkowski spacetime can be performed without invoking unphysical singularities in amplitudes. The high energy behavior of amplitudes is found to be regular at infinity provided that only space-space non-anticommutativity is allowed.

## 1 Introduction

In previous work [1], we developed a superspace fomalism based on coordinates

$$\rho^\mu = x^\mu + \beta^\mu, \tag{1}$$

where $x^\mu$ denote the familiar spacetime coordinates saisfying

$$[x^\mu, x^\nu] = 0, \tag{2}$$

and $\beta^\mu$ are coordinates in an associative Grassman algebra which satisfy

$$\{\beta^\mu, \beta^\nu\} \equiv \beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 0. \tag{3}$$

Here, $\mu = 0...3$ although the formalism can easily be extended to higher-dimensional spaces.

The product of two operators $\hat{\phi}_1$ and $\hat{\phi}_2$ in our superspace is given by the $\circ$-product

$$(\hat{\phi}_1 \circ \hat{\phi}_2)(\rho) = \left[\exp\left(\frac{1}{2}\omega^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\frac{\partial}{\partial\eta^\nu}\right)\phi_1(\rho)\phi_2(\eta)\right]_{\rho=\eta}$$
$$= \phi_1(\rho)\phi_2(\rho) + \frac{1}{2}\omega^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\phi_1(\rho)\frac{\partial}{\partial\rho^\nu}\phi_2(\rho) + O(\omega^2), \tag{4}$$

where $\omega^{\mu\nu}$ is a nonsymmetric tensor

$$\omega^{\mu\nu} = -\tau^{\mu\nu} + i\theta^{\mu\nu}, \tag{5}$$

with $\tau^{\mu\nu} = \tau^{\nu\mu}$ and $\theta^{\mu\nu} = -\theta^{\nu\mu}$. Moreover, $\omega^{\mu\nu}$ is Hermitian symmetric $\omega^{\mu\nu} = \omega^{\dagger\mu\nu}$, where $\dagger$ denotes Hermitian conjugation.

Let us define the notation

$$[\phi_1(\rho), \phi_2(\rho)]_\circ \equiv \phi_1(\rho) \circ \phi_2(\rho) - \phi_2(\rho) \circ \phi_1(\rho), \tag{6}$$

and

$$\{\phi_1(\rho), \phi_2(\rho)\}_\circ \equiv \phi_1(\rho) \circ \phi_2(\rho) + \phi_2(\rho) \circ \phi_1(\rho). \tag{7}$$

From (1) and (4), we obtain for the superspace operator $\hat{\rho}$:

$$[\hat{\rho}^\mu, \hat{\rho}^\nu]_\circ = 2\beta^\mu\beta^\nu + i\theta^{\mu\nu} + O(\theta^2), \tag{8}$$

$$\{\hat{\rho}^\mu, \hat{\rho}^\nu\}_\circ = 2x^\mu x^\nu + 2(x^\mu\beta^\nu + x^\nu\beta^\mu) - \tau^{\mu\nu} + O(\tau^2). \tag{9}$$

If we now choose the limit in which $\beta^\mu \to 0$ and $\tau^{\mu\nu} \to 0$, then we get from (8) and (9):

$$[\hat{\rho}^\mu, \hat{\rho}^\nu]_\circ \to [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{10}$$

$$\{\hat{\rho}^\mu, \hat{\rho}^\nu\}_\circ \to \{\hat{x}^\mu, \hat{x}^\nu\} = 2x^\mu x^\nu. \tag{11}$$

We see that (10) and (11) give us back the usual noncommutative expressions for the coordinate operators $\hat{x}^\mu$.

In the limit $x^\mu \to 0$ and $\theta^{\mu\nu} \to 0$, we get from (8) and (9):

$$[\hat{\rho}^\mu, \hat{\rho}^\nu]_\circ \to [\hat{\beta}^\mu, \hat{\beta}^\nu] = 2\beta^\mu\beta^\nu, \tag{12}$$

$$\{\hat{\rho}^\mu, \hat{\rho}^\nu\}_\circ \to \{\hat{\beta}^\mu, \hat{\beta}^\nu\} = -\tau^{\mu\nu}. \tag{13}$$

In the following, we shall the consider the simpler geometry determined by $\theta^{\mu\nu} = 0$. We define a $\diamondsuit$-product

$$(\hat{\phi}_1 \diamondsuit \hat{\phi}_2)(\rho) = \left[\exp\left(-\frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\frac{\partial}{\partial\eta^\nu}\right)\phi_1(\rho)\phi_2(\eta)\right]_{\rho=\eta}$$

$$= \phi_1(\rho)\phi_2(\rho) - \frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\phi_1(\rho)\frac{\partial}{\partial\rho^\nu}\phi_2(\rho) + O(\tau^2). \tag{14}$$

We now have

$$\{\hat{\rho}^\mu, \hat{\rho}^\nu\}_\diamondsuit = 2x^\mu x^\nu + 2(x^\mu\beta^\nu + x^\nu\beta^\mu) - \tau^{\mu\nu}. \tag{15}$$

A calculation of the one loop diagram in scalar field theory is finite, and the higher order loops will be finite to all orders of perturbation theory, due to the convergence of the modified Feynman propagator $\bar{\Delta}_F$ in momentum space [1].

We assume that there exists a classical limit as $\ell \to 0$, $\beta^\mu \to 0$ and $\tau^{\mu\nu} \to 0$, where $\ell$ is a natural unit of length, so that we obtain the classical c-number spacetime continuum with $\{\hat{x}^\mu, \hat{x}^\nu\} = 2x^\mu x^\nu$.

Unitarity of the S-matrix and crossing symmetry are two of the most important features of quantum field theory. In the following, we shall explore the consequences of unitarity for amplitudes in non-anticommutative scalar field theory, the positions of singularities in the amplitudes and the asymptotic properties of amplitudes. We consider the case when $\tau^{mn} \neq 0$ and $\tau^{00} = \tau^{0n} = 0 \quad (m, n = 1, 2, 3)$, because only then can we define a conjugate momentum operator and avoid difficulties with acausal dynamics. Moreover, we shall find for this case that the amplitudes have regular behavior at infinity.

# 2 Non-anticommutative Field Theory

We have for two operators $\hat{f}$ and $\hat{g}$ in our superspace manifold:

$$(\hat{f} \diamondsuit \hat{g})(\rho) = \frac{1}{(2\pi)^8} \int d^4k d^4q \tilde{f}(k) \tilde{g}(q) \exp\left[\frac{1}{2}(k\tau q)\right] \exp[i(k+q)\rho], \tag{16}$$

where

$$\tilde{f}(k) = \frac{1}{(2\pi)^4} \int d^4\rho \exp(-ik\rho) f(\rho), \tag{17}$$

and $(k\tau q) \equiv k_\mu \tau^{\mu\nu} q_\nu$. We shall assume that derivatives act trivially in our superspace

$$[\rho_\mu, \partial_\nu] = -\eta_{\mu\nu}, \quad [\partial_\mu, \partial_\nu] = 0, \tag{18}$$

where $\partial_\mu = \partial / \partial \rho_\mu$.

In contrast to the noncommutative field theories, the kinetic energy component of the action in non-anticommutative field theories is not trivially the same as the commutative field theories, because of the symmetry of the tensor $\tau^{\mu\nu}$. The action for the scalar field $\phi$ is

$$I = \int d^4\rho \Big[\frac{1}{2} \partial_\mu \phi(\rho) \diamondsuit \partial^\mu \phi(\rho) - \frac{m^2}{2} \phi(\rho) \diamondsuit \phi(\rho) - V_\diamondsuit(\phi)\Big], \tag{19}$$

where $V_\diamondsuit(\phi)$ is the scalar field potential. For our calculations we shall choose the potential:

$$V_\diamondsuit(\phi) = \frac{\lambda}{4!}\phi^\beta_\diamondsuit, \tag{20}$$

where $\phi^\beta_\diamondsuit$ denotes the $\beta$th order product of the field $\phi$ using the $\diamondsuit$-product. The commutative theories $\beta = 3$ and $\beta = 4$ give the well-known renormalizable theories, while for $\beta \geq 5$ the commutative theory is non-renormalizable.

In the noncommutative and the non-anticommutative cases, there is an ambiguity in applying the quantization procedure in position space. The usual quantization conditions are defined for the scalar field $\phi$ and its conjugate momentum $\pi$ at different spacetime points, while the $\star$ and $\diamondsuit$-products only make sense when the products are computed at the same spacetime point. We can avoid this problem by working only in momentum space when performing Feynman rule calculations to obtain matrix elements. We formally define a $\triangle$-product of field operators $\hat{f}(\rho)$ and $\hat{g}(\eta)$ at different superspace points $\rho$ and $\eta$:

$$\hat{f}(\rho) \triangle \hat{g}(\eta) = \exp\left(-\frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\frac{\partial}{\partial\eta^\nu}\right)f(\rho)g(\eta). \tag{21}$$

This product reduces to the $\diamondsuit$-product of $\hat{f}$ and $\hat{g}$ in the limit $\rho \to \eta$.

Let us now consider the Feynman rules for the non-anticommutative scalar field theory in our superspace with coordinates $\rho^\mu$. The modified Feynman propagator $\bar{\Delta}_F$ is defined by the vacuum expectation value of the time-ordered $\triangle$-product [1].

$$\begin{aligned} i\bar{\Delta}_F(\rho - \eta) &\equiv \langle 0|T(\phi(\rho) \triangle \phi(\eta))|0\rangle \\ &= \frac{i}{(2\pi)^4}\int \frac{d^4k \exp[-ik(\rho-\eta)]\exp[\frac{1}{2}(k\tau k)]}{k^2 - m^2 + i\epsilon}. \end{aligned} \tag{22}$$

In momentum space this gives

$$i\bar{\Delta}_F(k) = \frac{i\exp[\frac{1}{2}(k\tau k)]}{k^2 - m^2 + i\epsilon}, \tag{23}$$

which reduces to the standard commutative field theory form for the Feynman propagator

$$i\Delta_F(k) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{24}$$

in the limit $|\tau^{\mu\nu}| \to 0$.

From the interaction part of the action for the theory with $\beta = 4$:

$$I_{\rm int} = \frac{\lambda}{4!}\int d^4\rho\phi\diamondsuit\phi\diamondsuit\phi\diamondsuit\phi(\rho), \tag{25}$$

we can deduce that the vertex factor for the scalar non-anticommutative field theory has the form

$$V(k_1, k_2, k_3, k_4) = \exp\left\{\frac{1}{2}[(k_1\tau k_2) + (k_2\tau k_3) + (k_3\tau k_4)]\right\}. \tag{26}$$

The vertex function factor for a general diagram is given by

$$V(k_1, ..., k_n) = \sum_{i<j}\exp\left[\frac{1}{2}(k_i\cdot k_j)\right], \tag{27}$$

where $k_i\cdot k_j = k_{i\mu}\tau^{\mu\nu}k_{j\nu}$.

Our Feynman rules are: for every internal line we insert a modified Feynman propagator $\bar{\Delta}_F(k)$ and integrate over $k$ with the appropriate numerical factor. We associate with every diagram a vertex factor $V(p_1, ..., p_n; k_1, ..., k_n)$ where the $ps$ and $ks$ denote the external and internal momenta of the diagram, respectively.

In the non-anticommutative scalar field theory, the one loop self-energy diagram is *finite* for a generic value of the external momentum and for fixed finite values of an energy scale parameter $\Lambda$ in Euclidean momentum space. The same holds true for the vertex one loop corrections. The convergence of both planar and non-planar loop diagrams should hold to all orders of perturbation theory, because of the strong convergence of the modified Feynman propagator $\bar{\Delta}_F$ in Euclidean momentum space.

# 3 Unitarity of Amplitudes

A basic feature of quantum field theory is unitarity, which states that the S-matrix must satisfy the condition $SS^\dagger = 1$. In terms of the $T$ matrix defined by $S = 1 + iT$, we have

$$\frac{1}{2i}(T_{fi} - T^\dagger_{if}) = \frac{1}{2}\sum_n T^\dagger_{nf}T_{ni}. \tag{28}$$

Inserting momentum conservation

$$\langle f|T|i\rangle = (2\pi)^4\delta^4(P_f - P_i)\mathcal{T}_{fi} \tag{29}$$

we obtain

$$\frac{1}{2i}(\mathcal{T}_{fi} - \mathcal{T}^\dagger_{if}) = \frac{1}{2}\sum_n(2\pi)^4\delta^4(P_n - P_i)\mathcal{T}^\dagger_{nf}\mathcal{T}_{ni}. \tag{30}$$

In general, a scattering amplitude in non-anticommutative scalar field theory will involve the products of the modified Feynman propagator $\bar{\Delta}_f(k^2)$ for each internal line of a diagram with Euclidean momentum $k$ and vertex functions $V(k_1, ..., k_n; p_1, ..., p_n)$. Then, the amplitude describing a given diagram with $n$ external Euclidean momenta $q_j$, which satisfies the conservation law $q_1 + q_2 + ... + q_n = 0$, will be of the form

$$A = \int ... \int \prod_i d^4k_i \prod_j \frac{W_j(k_j^2, k_j q_j)}{k_j^2 + m_j^2}, \tag{31}$$

where $k_j$ is an Euclidean 4-momentum and $m_j$ is the mass of the jth particle. The integration is carried out over the internal line momentum $k_j$. Moreover, $W(k_j^2, k_j q_j)$ is an *entire* function in the complex $k$ plane and decreases rapidly as Re$k \to +\infty$. For our non-anticommutative field theory, the asymptotic behaviour $k_j^2 \to \infty$ is

$$W_j(k_j^2, k_j q_j) \sim \exp(-a k_j^2/\Lambda^2), \tag{32}$$

where $a$ is a constant. When the natural unit of length $\ell \to 0$, yielding the classical spacetime continuum, it is understood that this corresponds to $\Lambda \to \infty$.

Apart from a constant factor, the Euclidean amplitude $A$ should coincide with the real amplitude corresponding to a region with $p_i p_j = -q_i q_j$, where the $p_i$ and $q_i$ denote the momenta in the physical region and in Euclidean momentum space, respectively. By an analytic continuation, we can obtain the amplitude in the physical region of the invariant external momenta $p_i$ with the understanding that the masses involve an additional negative imaginary part: $m_i \to m_i - i\epsilon$. When $W(k_i, q_j) = 1$ or is just a polynomial with respect to $k$, then $A$ coincides with the real Minkowski amplitude of standard local field theory, and the Minkowski and Euclidean formulations are completely equivalent [2].

The analysis by Landau [3] of the singularities of amplitudes in perturbation theory is based on the amplitudes in Euclidean space. In the non-anticommutative field theories, we must determine the singularities of the amplitude with respect to the invariant momentum variables. We can rewrite the amplitude using the Feynman parameterization

$$A = (N-1)! \int_0^1 ... \int_0^1 d\alpha_1 ... d\alpha_n \delta(1 - \sum_{i=1}^n \alpha_i) \int ...$$

$$\times \int \prod_i d^4k_i \prod_j \frac{W_j(k_j^2)}{\sum_j \alpha_j (k_j^2 + m_j^2)}, \tag{33}$$

where $N$ is the number of internal lines. Following Landau [3], we obtain

$$\sum_{i=1}^{n} \alpha_i (k_i^2 + m_i^2) = \chi(\alpha, q_i q_j, m_j^2) + K(\alpha, k'), \tag{34}$$

where $\chi$ is a non-homogeneous quadratic form in the momenta $q_i$ that describes the free ends of the diagram, and $K$ is a homogeneous quadratic form in the integration variable $k'$ with coefficients that only depend on the parameters $\alpha_i$.

Since the numerator of the amplitude contains an entire function of the scalar products $q_i q_j$ and the parameters $\alpha_i$, it cannot lead to any additional singularities in a finite region of the invariant momentum variables. However, to determine whether our amplitude satisfies the Cutkosky cutting rules [4], and thereby the unitarity relation for the S-matrix, we must calculate the discontinuities of $A$ across the appropriate branch cuts.

We decompose the amplitude $A$ into two parts $A_I$ and $A_{II}$ connected by $r$ internal lines as shown in Fig 1.

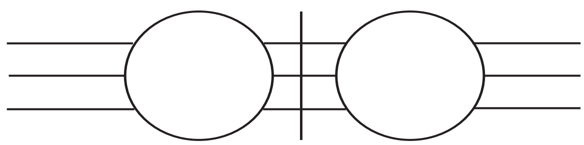

Fig. 1.

This gives

$$A = \int ... \int dk_1 ... dk_r A_I(q_j, k_i) \prod_{n=1}^{r} \frac{W_n(k_n^2, q)}{k_n^2 + m_n^2} A_{II}(q'_j, k_i) \delta^4(q - k_1 - ... - k_r), \tag{35}$$

where $q_j$ and $q'_j$ denote the external momenta corresponding to the parts I and II, respectively. The amplitude considered as a function of the variable $z = -q^2$ has a branch cut starting at the point $z = (m_1 + ... + m_r)^2$ and the discontinuity of $A$ is given by

$$\Delta A(z) = i(2\pi)^r \prod_{n=1}^{r} W_n(-m_r^2, q) \int d^4\tilde{k}_1 ... \int d^4\tilde{k}_r \prod_{n=1}^{r} \theta(\tilde{k}_{n0}) \delta(\tilde{k}_n^2 + m_n^2)$$

$$\times \delta^4(\tilde{q} - \tilde{k}_1 - ... - \tilde{k}_r) A_I(q_j, \tilde{k}_i) A_{II}(q'_j, \tilde{k}_i). \tag{36}$$

Here, $\tilde{k}_i$ denotes 4-vectors with the components $(\mathbf{k}, ik_{i0})$, with $\tilde{k}_i^2 = \mathbf{k}_i^2 - k_{i0}^2$, $(\tilde{k}_i q_j) = \mathbf{k}_i\mathbf{q}_j + i\mathbf{k}_{i0}q_{j4}$. The vector $q$ with components $(\mathbf{q}, iq_0)$ satisfies $q^2 = \mathbf{q}^2 - q_0^2 = -z$.

The result (36) is known as the Cutkosky cutting rule for normal thresholds. It is satisfied by standard, local field theory perturbative amplitudes, and can be extended to anomalous thresholds as well. The proof that it is satisfied for amplitudes containing entire functions $W$ was given by Efimov [5], in the context of a nonlocal field theory, using the methods of Landau [3] and Rudik and Simonov [6]. The proof consists in showing that it is possible to analytically continue the amplitude $A(z)$ from the region $z < 0$ to the physical region $z > 0$ and to reveal the singularities of the amplitude, which depend only on the intermediate momentum lines of the diagram. The derivation of the proof for two internal lines leads to the result

$$\Delta A(z) = -(2\pi)^2\theta[z - (m_1 + m_2)^2] W_1(-m_1^2, z) W_2(-m_2^2, z) C(-m_1^2, -m_2^2, z), \tag{37}$$

where

$$C(-m_1^2, -m_2^2, z) = \frac{-i[-u(m_1^2, m_2^2, z)]^{1/2}}{8z}$$
$$\times \int d\Omega_n \Phi\Big\{\mathbf{n}\Big[\frac{-u(m_1^2, m_2^2, z)}{4z}\Big]^{1/2} \frac{i(m_1^2 - m_2^2 + z)}{2\sqrt{z}}\Big\}. \tag{38}$$

Moreover, $z = -q^2$ and

$$u(a, b, c) = 2ab + 2bc + 2ca - a^2 - b^2 - c^2 = [(\sqrt{a} + \sqrt{b})^2 - c][c - (\sqrt{a} - \sqrt{b})^2]. \tag{39}$$

The integration in (38) is performed with respect to all the directions of the three-dimensional unit vector $\mathbf{n}$, and the $\tilde{q}$ vector has only a fourth component ($\mathbf{q} = 0, q_4 = \sqrt{z}$). The proof of (36) can be extended to diagrams with an arbitrary number of internal lines.

It follows from these results that amplitudes calculated in the non-anticommutative scalar field theory will not change the analyticity properties of the theory in any finite region of the momentum variables, and the validity of the Cutkosky cutting rules leads to a proof of the unitarity of this field theory. The singularities of the amplitude in the momentum variables come only from the zeros of the denominator. From this, it follows that the Landau and Cutkosky analyses can be applied and all necessary conditions for the fulfillment of unitarity are satisfied. Landau required that the amplitudes were regular at infinity in the momentum plane. He was then able to perform a Wick contour rotation ($k_0 \to ik_4$) and relate the zeros of a diagram

to the zeros of a known quadratic form in the external momenta subsequent to an application of the Feynman $\alpha$ parameterization.

In Euclidean momentum space, our amplitudes are not affected by possible essential singularities at infinity and the positions of singularities of diagrams are likewise not affected. When we analytically continue to Minkowski spacetime, all integrals converge and the amplitudes are regular at infinity for $\tau^{00} = \tau^{0n} = 0$ and $\tau^{mn} \neq 0 \quad (m, n = 1, 2, 3)$. This property of the amplitudes will be proved in the next Section.

# 4 Asymptotic Behavior of Amplitudes

We shall now investigate the high energy behavior of amplitudes. Let us consider the second order amplitudes described by diagrams of the type Fig. 2.

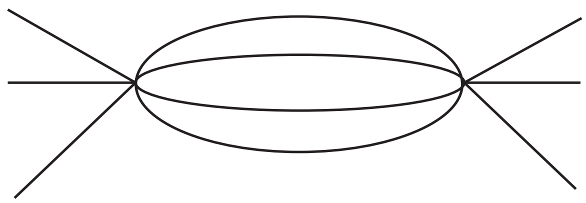

Fig. 2

There are $r$ internal lines and $n$ external lines meet at each vertex. We denote by $p$ the sum of the $n$ external momenta. The amplitude corresponding to this graph has the form

$$A_2(p^2) = \lambda^2 \int d^4\rho \exp(ip\rho)\bar{\Delta}^r_{mF}(\rho)V^r(\rho) = \lambda^2 \int d^4\rho \exp(ip\rho)W^r(\rho), \tag{40}$$

where $\bar{\Delta}^r_{mF}$ is the modified causal propagator for a mass $m$ and $r$ internal lines, $V^r(\rho)$ is the non-anticommutative vertex factor and the integral is taken over the four-dimensional superspace coordinates.

We use the relation

$$\bar{\Delta}^r_{mF}(\rho) = \int_{(mr)^2}^{\infty} d\kappa^2 \Omega^r(\kappa^2)\bar{\Delta}_{\kappa F}(\rho), \tag{41}$$

where $\Omega^r(\kappa^2)$ is the phase volume of $r$ scalar particles with mass $m$:

$$\Omega^r(\kappa^2) = \frac{1}{(2\pi)^{3(r-1)}} \int \frac{d\mathbf{k}_1}{2\omega_1} ... \int \frac{d\mathbf{k}_r}{2\omega_r} \delta^4(k - k_1 - ... - k_r), \tag{42}$$

where $\omega_i = \sqrt{\mathbf{k}_i^2 + m^2}$ and on shell $k^2 = k_0^2 - \mathbf{k}^2 = \kappa^2$.

We now have

$$\begin{aligned} A_2(p^2) &= \lambda^2 \int_{(mr)^2}^{\infty} d\kappa^2 \Omega^r(\kappa^2) \bar{\Delta}_\kappa(p^2) V_\kappa(p) \\ &= \lambda^2 \int_{(mr)^2}^{\infty} d\kappa^2 \Omega^r(\kappa^2) \Big[ \frac{B_\kappa(p)}{-p^2 + \kappa^2 - i\epsilon} \Big], \end{aligned} \tag{43}$$

where

$$B_\kappa(p) = \exp\Big[\frac{1}{2}(p\tau p)\Big] V_\kappa(p). \tag{44}$$

Writing $s = p^2$, the amplitude $A_2(s)$ is real in the region $s < (rm)^2$ and vanishes for $s \to -\infty$. In the complex s-plane, it has a branch cut beginning at $s = (rm)^2$, and the imaginary part is

$$\mathrm{Im} A_2(s) = \pi \lambda^2 \Omega^r(s) B(\kappa^2) \tag{45}$$

which is required by unitarity. We also have

$$\mathrm{Re} A_2(s) = \lambda^2 \int_{(mr)^2}^{\infty} d\kappa^2 \Omega^r(\kappa^2) \Big[ \frac{B_\kappa(p)}{-s + \kappa^2} \Big]. \tag{46}$$

The dominant behavior of $B_\kappa(p)$ as $p^2 \to +\infty$ is

$$B_\kappa(p) \sim \exp[a(p\tau p)], \tag{47}$$

where $a$ is a positive constant and $(p\tau p) = p_\mu \tau^{\mu\nu} p_\nu$. Let us choose an orthonormal frame such that $\tau^{\mu\nu} = \eta^{\mu\nu}/\Lambda^2$ where $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. Then, for $p^2 \to +\infty$, we have

$$B_\kappa(p) \sim \exp\Big(ap^2/\Lambda^2\Big), \tag{48}$$

and from (46), it follows that $\mathrm{Re} A_2(s)$ increases more rapidly than any finite power of $s$ as $s \to +\infty$.

Let us instead choose an orthonormal frame with $\tau^{00} = \tau^{0n} = 0$ and $\tau^{mn} = -\delta_{mn}/\Lambda^2 \quad (m, n = 1, 2, 3)$. Then, we have

$$B_\kappa(p) \sim \exp\Big(-a\mathbf{p}^2/\Lambda^2\Big). \tag{49}$$

Since $\mathbf{p}^2 > 0$, it follows that for $s \to \pm\infty$ (and $\mathbf{p}^2 \to +\infty$), $\mathrm{Re}A_2(s)$ vanishes rapidly for $s \gg \Lambda^2$.

If we allow $\tau^{00} \neq 0$ and $\tau^{0i} \neq 0$, then $\mathrm{Re}A_2(s)$ will have an essential singularity at $s \to +\infty$. While this will not violate unitarity, because there are no additional singularities in the finite complex $s$-plane, it will lead to unphysical behavior of the scattering amplitudes at high energies and unphysical crossing symmetry relations. However, our choice of $\tau^{\mu\nu}$ *breaks Lorentz invariance*. It was argued by Efimov [5] that a summation of a certain class of graphs can correct the high energy behavior problem of the amplitudes. But there is no convincing proof that the unphysical increase of amplitudes can be compensated for by the inclusion of higher order graphs.

# 5 Conclusions

We have investigated the Cutkosky rules and unitarity in a non-anticommutative scalar field theory. For arbitrary Feynman diagrams in Euclidean space, the Cutkosky rules are valid for normal thresholds. There are no additional unphysical singularities present in the complex momentum plane and the transition to the physical region of external momenta can be achieved by analytic continuation of the amplitudes with respect to the invariant momenta. The modified Feynman propagator $\bar{\Delta}_F(p^2)$ and the vertex factor $V(k,p)$ are entire functions of the momenta, which allows an analysis of the singularity structure of the amplitudes in perturbation theory in the manner of Landau and Efimov. The results lead to a proof of unitarity.

The high energy behavior of scattering amplitudes was studied and it was found that crossing symmetry relations and high energy behavior can be physical, provided that we restrict ourselves to the case $\tau^{mn} \neq 0$ and $\tau^{00} = \tau^{0n} = 0$. The same circumstances exist for noncommutative field theories in which $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$, because the restriction that $\theta^{mn} \neq 0$ and $\theta^{0n} = 0$ avoids problems with unitarity and causality. We can avoid a 'hard' breaking of Lorentz invariance by invoking a spontaneous breaking of Lorentz invariance [7, 8]. In this scenario, a Higgs vector symmetry breaking mechanism is introduced which allows a soft breaking of $SO(3,1)$ to $O(3)$ at short distance scales when the non-anticommutative natural unit of length $\ell \neq 0$ and $|\tau^{\mu\nu}| \neq 0$.

**Acknowledgments**

I thank Michael Clayton for helpful discussions. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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